A Time and Space Routing Game Model applied to Visibility Competition on Online Social Networks

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Abstract— **Companies or users that want to do advertising on Online Social Networks, need to know in which News Feed it would be more efficient to post, at what time of day the posts would have more visibility and which topic of message they should choose to reach popularity. We propose to answer these questions with the help of the Routing Game Theory, which considers a finite number of players and splittable demands. First, we propose a visibility measure on a News Feed. Next, we present our game based on visibility measures, which is similar to the Weighted Allocation Game. Following this consideration, we provide the uniqueness of the Nash Equilibrium, a characterization via concave programming, and its closed form. Then, we propose a two time scale decentralized algorithm where each user estimates the total flow of messages in each News Feed and uses stochastic gradient algorithm to update their own flows. Finally, we use a real data set in order to estimate the parameters of our model. The goal of this model is to provide to the Online Social Network Authority a better knowledge of posting behavior of competitive users.**

I. Introduction

Lately, new questions concerning Online Social Networks and the visibility of its content are attracting the attention of researchers of this field of study. For each user or company that wants to become popular and visible in Social Network, answers to the following questions seems to be essential: Where and when should it be posted? On what topic? How many messages should it post? In this paper we consider that these questions have to be treated following a competitive scenario, since there are several users/companies aiming to become popular in Social Networks. Also, the Routing Game Theory seems to be an interesting tool to answer these questions. So far, to our knowledge, only one paper has adopted the Routing Game Theory to solve problems concerning Social Networks [23].

The Routing Game Theory supposes a game where its players have to decide where they send their flows of data. In this sense, the player's flow has the *splittable property*. Another property of Routing Games is the *congestion property*. Mathematically speaking, this property can be explained by the fact that the utility of a player on a link decreases in the total amount of flow sends in this link. This congestion property can also be found in Online Social Network when the visibility of a message decreases on the total flow of messages. One of the fundamental papers in Routing Game is [3].

This paper is an extension of [4]. In [4] the sources control the flow of message sends in a News Feed. So far, in the case of Facebook, a *News Feed* is a Feed where the user's friends and subscribed pages' news are published. These posts are displayed in chronological order from the oldest to the newest. It is possible to find in the introduction of [24] more definitions of a News Feed. Content generators (in this paper, we will call them *sources*) want to have the maximum number of messages on the top of the News Feed for each time they post. If we take this goal as a measure of visibility for a *source* on a News Feed, then probably this measure decreases in the sum of messages in the News Feed, a phenomena that can be observed in a Routing Game. This model was first studied in [4], where its authors prove that it can be comparable to the well-known *Rent seeking Game* [10].

In section II, we will present the latest related works concerning Game Theory applied to competition over visibility in Online Social Networks. Then, in the case of our model, we will present mathematical models that are similar to ours.

In this paper, we consider that sources can post in several News Feeds, at different times per day, and concerning different topics of messages. In section III, we propose a visibility measure and also we develop the routing game theory associated.

There is a contrast between the Routing Games defined in this paper and classical Routing Games [3]. This why we felt the need to adapt the theory of [3] to the Routing Game defined in this paper. We will develop it in section IV. First, we prove the existence of the Nash Equilibrium. Then, we prove that at the Nash Equilibrium, under some hypothesis, a source will only send messages about a unique topic per News Feed. Moreover, we give a characterization of the Nash Equilibria based on a Concave Programming and we prove its uniqueness.

Section V deals with more practical issues. Indeed, the Routing Game Theory developed in [4] provides existence and uniqueness results about the Nash Equilibrium. However, it does not propose realistic decentralized learning algorithm. A property that allows us to have some mechanism that converges to a Nash Equilibrium is the *potential property*. If we define $s_j \in [0,1]$ the strategy of a player j, $j \in \{1,\ldots,J\}$, and $C_j(s_1,\ldots,s_J)$ his payoff, then under differentiability assumption on the player's payoff, $P(s_1,\ldots,s_J)$ is a *potential*

This work was supported by the European project CONGAS

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for this game if $\frac{\partial C_j}{\partial s_j}(s_1,\dots,s_J)=\frac{\partial P_j}{\partial s_j}(s_1,\dots,s_J)$ for all j and all s_j. Nevertheless, in the case of a Routing Game of [3] or a News Feed Game [4], it can be proved that the previous property is not satisfying if applied to our model. Despite this and based on the results of IV , we are able to provide the convergence of a decentralized mechanism to the Nash Equilibrium.

In section VI, we propose a method that allows us to apply our model to real data. Besides that, we propose a few results concerning the estimation of the parameters of our game.

II. Related Works

A. Related Models

- Studies concerning users/companies that want to become visible on News Feeds have only been treated in [4] and [24]. This is not the usual way of modeling competition over visibility on Online Social Networks. There are two common ways : the first one is the advertising allocation. In this case, sources can choose to pay to Online Social Network to get their content promoted. Papers about this subject are often linked to marketing differential game ([14], [30], [11], [12], [13]). The second one is to consider an epidemic process on a Social Network. In this case, the question would be which node a source need to contaminate in order to maximize the spread of its contents ([31], [32], [33]).

In [24] the authors use the Poisson process to model the flow of messages on a unique News Feed. However, it can be expected that the Poisson process does not fit with realistic scenario. That's the reason why we propose to model the flow of messages of each source by a Stationary Point Process, as it is proposed in [4]. This model would allow sources to post at different times per day, on different News Feeds and concerning different topics.

B. Related Mathematical Results

Under some hypothesis, and from the fact that at equilibrium player's messages concerns only one topic per News Feed, the game that we study in this paper is quasi-equivalent to the game defined in [25]. This game is called *Weighted Allocation Game*. The authors prove the existence of the Nash Equilibrium. Also, they study efficiency measurements and they provide learning algorithms concerning the Nash Equilibrium. In our case, we prove the uniqueness and the characterization of the equilibrium based on the Routing Game Theory. Concerning the learning algorithm, considerable works have been done to provide learning algorithm in Allocation Games ([25], [26] [27], [28], [29]) and in Routing Games ([3], [6], [8], [9]). However, our algorithm is based on real scenarios and has not been studied yet.

III. Model

A. Creation of a Visibility Measure on a News Feed

In this section, we propose to define a measure of visibility of a source j, $j\in\{1,\dots,J\}$ in a News Feed. A News Feed in an Online Social Network is a continuous transmission of data in the user's personal account. The content of this data comes from other users and pages that the user chooses to be subscribed to. We consider that the messages in a News Feed of a user are ordered from the newest to the oldest. This sort of News Feed configuration can be found in the most popular social networks, like Twitter, Facebook or Tumblr. In order to model the visibility of a content in a News Feed, we will use the theory of Point Processes [15]. As a first step, to each source j we associate a N_j stationary point process which represents the messages sent by j in the News Feed of a user m. With the help of this model, a first visibility measure of the content of j will be proposed. As a second step, we will relax the hypothesis of stationarity of each N_j by cutting the time in peak intervals. Then, we will define a new process for each peak interval in which a new measure of visibility will be proposed. Finally, as a third step, we will distinguish messages by topic and our final measure of visibility of a source j will be given.

Visibility measure 1.0: Since most web users give more attention to the information above the page fold [19], we focus our visibility measure on the first message of the News Feed. The arrival time of any message in the News Feed of a user m is modeled by a stationary point process (N,θ_t,P), where θ_t is the associated flow, P the associated probability measure and $\lambda := E[N(0,1]]$ the intensity of the process with $0<\lambda<\infty$. T_n is the arrival time of n-th message. Let's define the sequence of mark $\{Z_n\}_n$ associated to (N,θ_t,P) with $Z_n\in\{1,\dots,J\}$ for all n. Z_n tell us the origin of the n-th message. Let's define the continuous process $Z(t)$ where $Z(t)=Z_n$ if $t\in(T_n,T_{n+1}]$. Let's define a new process N_j:

$$N_j((a,b]) := \sum_{n\in\mathbb{Z}} 1_{\{Z_n=j\}} 1_{\{T_n\in(a,b]\}}. \tag{1}$$

This process counts the number of messages from j during $(a,b]$. We assume that the intensity of this new process is λ_j, with $0<\lambda_j<\infty$. We consider that the visibility measure of a source j at time t, is

$$\lim_{h\to 0} P(Z(T_+(t))=j \mid N(t,t+h]\geq 1) \tag{2}$$

or $T_+(t)=t+inf\{h \mid N(t,t+h)=1\}$.

This measure is the probability that, when a message arrives in the News Feed of m, it will be originated from source j. The next proposition gives an explicit formula for this probability.

Lemma 1: For all t and for all j,

$$\lim_{h\to 0} P(Z(T_+(t))=j \mid N(t,t+h]\geq 1)=\frac{\lambda_j}{\sum_i \lambda_i}. \tag{3}$$

Proof: Let $j\in\{1,\dots,J\}$ and $t\in\mathbb{R}$,

$$P_N^0(Z(0)=j)=\lim_{h\to 0} P(Z(T_+(t))=j \mid N(t,t+h]\geq 1)$$

where $P_N^0(Z(0)=j)=\frac{1}{\lambda(b-a)}E\left[\sum_{n\in\mathbb{Z}} 1_{Z_n=j}1_{T_n\in(a,b]}\right]$. P_N^0 is called the Palm probability of the marks Z_n. The first equality comes from the local interpretation of Palm probability (p.40 [15]). From the definition of the intensity of N_i, it is possible to write:

$$\lambda_j(b-a)=E\left[\sum_{n\in\mathbb{Z}} 1_{Z_n=j}1_{T_n\in(a,b]}\right]=\lambda(b-a)P_N^0(Z(0)=j).$$

The first equality comes from the definition of the intensity of N_j and the second equality comes from the definition of $P_N^0(Z(0)=j)$. From the definition of N_j we have $\lambda = \sum_i \lambda_i$. Then,

$$P_N^0(Z(0)=j) = \frac{\lambda_j}{\sum_i \lambda_i}.$$

■

The fact that we restrict our measure to arrival times allows us to have an explicit form for any stationary point process. Moreover, the visibility measure of the source j gives the proportion of messages that originate from j in all the News Feed content. We cannot consider that all the messages sent by sources can be modeled by stationary point process. There are peak hours when a higher number of users are connected at the same time [16]. To consider this phenomenon, we propose to decompose our point process.

Visibility measure 2.0: Two examples can be used to contradict the stationarity of messages arrival processes in Social Network:

- Firstly, there are hours in a day when a higher number of users are connected at the same time [16]. If the source identifies these peak hours, and posts in this specific moment, it might increase the visibility of its content.
- Secondly, when a source is connected to a media, it seeks to create a real time content to generate interaction with its subscribers and increase the number of views of its content. Radio programs, for example, create content during the show and can even interact live with users.

To address the issue on stationarity, we define $[a_q, b_q]$, as being a peak interval, with $q \in \{1,\dots,Q\}$ and with $a_q < b_q$. A peak interval is a time interval when sources need to be visible. Thus we define the point process (N_q, θ_t, P_q) modeling the arrivals of messages on the News Feed m in $[a_q+T, b_q+T]$ for all $T \in \mathbb{R}$. We assume that $[a_q+T, b_q+T] \cap [a_{q'}+T', b_{q'}+T'] = \emptyset$ for all q, q', T, T'. We also assume that N_q et $N_{q'}$ are independent for all q and q'. By defining Z_q, $N_{(j,q)}$ and $\lambda_{(j,q)}$ as before, we can prove that:

$$\lim_{h\to 0} P(Z_q(T_+(t)) = j \mid N_q(t,t+h] \geq 1) = \frac{\lambda_{(j,q)}}{\sum_i \lambda_{(i,q)}}. \quad (4)$$

So far the influence of a source does not take into account the information of its messages. In the next measure, we will consider the topics treated in messages.

Final Visibility measure : We propose that for each peak interval $[a_q, b_q]$ a message can give information about a topic c, $c \in \{1,\dots,C\}$. We must therefore define for each q and for each c a point process $(N_{(c,q)}, \theta_t, P_{(c,q)})$ of intensity $\lambda_{(c,q)}$. We assume that each $(N_{(c,q)}, \theta_t, P_{(c,q)})$ and $(N_{(q,c')}, \theta_t, P_{(c,q)})$ are independent. This is why by defining $Z_{(c,q)}$, $N_{(j,c,q)}$ and $\lambda_{(j,q,c)}$ as previously, we can prove that:

$$\lim_{h\to 0} P(Z_{(q,c)}(T_+(t)) = j \mid N_q(t,t+h] \geq 1) = \frac{\lambda_{(j,c,q)}}{\sum_i \sum_{c'} \lambda_{(i,c',q)}}. \quad (5)$$

We consider that $V_{j,q,c} := \frac{\lambda_{(j,c,q)}}{\sum_i \sum_{c'} \lambda_{(i,c',q)}}$ is a measure of the visibility of j in the peak interval $[a_q, b_q]$ about topic c. This measure can take into account that a player j can decrease the visibility on a topic by increasing the visibility of another topic.

B. Game Model applied to Social Network

Due to the presence of many sources wishing to be visible in social networks news feeds, we propose to define a game between sources. Firstly, the strategies of sources will be described. We assume that sources can post on different News Feeds that are not interconnected between them. Secondly, the utility of each player will be given. We propose to define the utility of a player as the weighted sum of measures of visibility that is associated to him.

Source's Strategies: We assume that a source j can send messages to F News Feeds. The strategy vector of the source j is $\vec{\lambda}_j = (\lambda_{(j,q,c,f)})_{q,c,f}$, with $f \in \{1,\dots,F\}$. A component $\lambda_{(j,q,c,f)}$ of $\vec{\lambda}_j$ is the intensity of the message's flow sent by j on the topics c at the peak interval $[a_q, b_q]$ in News Feed f. In order to simplify the notations, we propose to define $l = (q,f)$ with $l \in \{1,\dots,L\}$ and $L = Q \times F$. This allows us to write, for each j, $\vec{\lambda}_j = (\lambda_{(j,c,l)})_{c,l}$. We also assume that:

- Firstly, if a source chooses to send a message to a specific News Feed, it cannot send the same message to a different News Feed.
- Secondly, a source can only send a limited number of messages per time unit.

Let us define for all j and c, $\lambda_{(j,c,0)}$ which is the message's flow about a topic c that the source j doesn't want to send to any News Feeds. The strategy vector of the source j becomes $\vec{\lambda}_j = (\lambda_{(j,c,l)}, \lambda_{(j,c,0)})_{c,l}$. Then the previous assumptions imply that for all j and for all c, $\sum_l \lambda_{(j,c,l)} + \lambda_{(j,c,0)} = \phi_{(j,c)}$, with $\phi_{(j,c)} > 0$.

After having defined the strategies of sources, we must now define their utilities.

Source's Utility: In order to define the utility of a source, we recall the visibility measure vector of j:

$$\vec{V}_j = \left(\frac{\lambda_{(j,c,l)}}{\sum_i \sum_c \lambda_{(i,c,l)} + \sigma_l}\right)_{c,l}, \quad (6)$$

where σ_l is a non-controlled flow on each link.

A source j wants to maximize $\vec{V}_j$. In order to achieve this, we propose to model the utility of j by

$$U_j(\vec{\lambda}_j, \vec{\lambda}_{-j}) = \sum_l \left[\frac{\sum_c \gamma_{(j,c,l)} \lambda_{(j,c,l)}}{\sum_i \sum_c \lambda_{(i,c,l)} + \sigma_l}\right] + \gamma_0 \sum_c \lambda_{(j,c,0)}, \quad (7)$$

Where $\sum_{l,c} \gamma_{(j,c,l)} + \gamma_0 = 1$. Thus the utility we have to define is a weighted sum of the measures visibilities of j. In the next section, our interest turns to studying the game that we defined previously.

IV. Premilinaries

In this section, we will study the properties of Nash Equilibria of the game. To start with, we will provide some characterizations of Nash Equilibria and also the existence of at least one of them. Next, we will prove that at any Nash Equilibrium, each player sends messages about a unique topic. At this point we will characterize Nash Equilibria via Concave Programming. Finally, the uniqueness of the Nash Equilibrium will be proved.

We are in a non-cooperative framework, where each source wants to maximize its visibility. Let's define $\vec{\sigma} = (\sigma_1, \ldots, \sigma_L)$. We called $\mathbf{G}_{\vec{\sigma}}$ our game, where each source j wants to maximize in $\vec{\lambda}_j$:

$$U_j(\cdot, \vec{\lambda}_{-j}) \quad (8)$$

$$\text{s.t. } \sum_l \lambda_{(j,1,l)} + \lambda_{(j,1,0)} = \phi_{(j,1)}, \ldots, \quad (9)$$

$$\textstyle\sum_l \lambda_{(j,C,l)} + \lambda_{(j,C,0)} = \phi_{(j,C)}, \quad (10)$$

$$\lambda_{(j,1,1)} \geq 0, \ldots, \lambda_{(j,C,L)} \geq 0. \quad (11)$$

The utility of a source j can be rewritten in the following form:

$$U_j(\vec{\lambda}_j, \vec{\lambda}_{-j}) = \sum_l \left[\frac{\sum_c \gamma_{(j,c,l)} \lambda_{(j,c,l)}}{\sum_i \sum_c \lambda_{(i,c,l)} + \sigma_l} \right] \quad (12)$$

$$+\gamma_0 \sum_c \left[\phi_{(j,c)} - \sum_l \lambda_{(j,c,l)} \right]. \quad (13)$$

with $\vec{\lambda}_j = (\lambda_{(j,c,l)})_{c,l}$. We suggest using the concept of Pure Nash Equilibrium (**PNE**) in order to solve the game:

Definition 2: $(\vec{\underline{\lambda}}_1, \ldots, \vec{\underline{\lambda}}_J)$ is a **PNE** iff for all j:

$$U_j(\vec{\underline{\lambda}}_j, \vec{\underline{\lambda}}_{-j}) = \max_{\vec{\lambda}_j} U_j(\vec{\lambda}_j, \vec{\underline{\lambda}}_{-j}). \quad (14)$$

We impose the following assumption:

(H1) It exists a **PNE** of $\mathbf{G}_{\vec{\sigma}}$ such that for all j, c,

$$\sum_l \underline{\lambda}_{(j,c,l)} < \phi_{(j,c)}.$$

The assumption **(H1)** characterizes situations where an equilibrium exists such that any source cannot saturate its constraints.

The existence of at least one **PNE** comes from the fact that for all j the function $U_j(\cdot, \vec{\lambda}_{-j})$ is concave in $\vec{\lambda}_j$ and from [5]. The existence being ensured, we need a mathematical characterization of **PNE** (KKT characterization) for their study:

Consider that **(H1)** is verified. $\vec{\underline{\lambda}} = \{\vec{\underline{\lambda}}_1, \ldots, \vec{\underline{\lambda}}_J\}$ is a **PNE** iff for all j and for all l,

$$\underline{\lambda}_{(j,c,l)} > 0 \quad \Rightarrow \quad \frac{\partial U_j}{\partial \lambda_{(j,c,l)}}(\vec{\underline{\lambda}}_j, \vec{\underline{\lambda}}_{-j}) = 0, \quad (15)$$

$$\underline{\lambda}_{(j,c,l)} = 0 \quad \Rightarrow \quad \frac{\partial U_j}{\partial \lambda_{(j,c,l)}}(\vec{\underline{\lambda}}_j, \vec{\underline{\lambda}}_{-j}) \leq 0,$$

This characterization is due to the fact that $U_i(\cdot, \lambda_{-j})$ is concave continuous differentiable in λ_j and because constraints are not saturated.

A. Only one topic per player per Social Network at Equilibria

In this section we will prove that if a source for a specific social network has not equal preferences on two different topics then it will only send messages that concern the unique topic that it prefers the most. It is summarized in the next proposition.

Proposition 3: Consider that $\gamma_{(j,c,l)} > \gamma_{(j,c',l)}$. Then for any $\underline{\lambda}_{(j,c,l)}$ and $\underline{\lambda}_{(j,c',l)}$ that verified **(H1)**,

$$\underline{\lambda}_{(j,c,l)} = 0, \ \underline{\lambda}_{(j,c',l)} = 0 \quad (16)$$

or

$$\underline{\lambda}_{(j,c,l)} > 0, \ \underline{\lambda}_{(j,c',l)} = 0. \quad (17)$$

Proof: Let j, l, c and c'. First let assume that $\underline{\lambda}_{(j,c',l)} > 0$. Then by (15), $\underline{\lambda}_{(j,c,l)}$ and $\underline{\lambda}_{(j,c',l)}$ verify:

$$\frac{\gamma_{(j,c,l)}}{\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l} - \frac{\sum_{c''} \gamma_{(j,c'',l)} \sum_{c''} \lambda_{(j,c'',l)}}{(\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l)^2} - \gamma_0 \leq 0, \quad (18)$$

$$\frac{\gamma_{(j,c',l)}}{\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l} - \frac{\sum_{c''} \gamma_{(j,c'',l)} \sum_{c''} \lambda_{(j,c'',l)}}{(\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l)^2} - \gamma_0 = 0, \quad (19)$$

which implies

$$\frac{\gamma_{(j,c,l)}}{\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l} - \frac{\sum_{c''} \gamma_{(j,c'',l)} \sum_{c''} \lambda_{(j,c'',l)}}{(\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l)^2} - \gamma_0 \quad (20)$$

$$\leq \frac{\gamma_{(j,c',l)}}{\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l} - \frac{\sum_{c''} \gamma_{(j,c'',l)} \sum_{c''} \lambda_{(j,c'',l)}}{(\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l)^2} - \gamma_0 \quad (21)$$

and we can deduce the following contradiction $\gamma_{(j,c,l)} \leq \gamma_{(j,c',l)}$. Then let's assume that $\underline{\lambda}_{(j,c',l)} > 0$ and $\underline{\lambda}_{(j,c',l)} > 0$ then the KKT condition implies

$$\frac{\gamma_{(j,c,l)}}{\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l} - \frac{\sum_{c''} \gamma_{(j,c'',l)} \sum_{c''} \lambda_{(j,c'',l)}}{(\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l)^2} - \gamma_0 \quad (22)$$

$$= \frac{\gamma_{(j,c',l)}}{\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l} - \frac{\sum_{c''} \gamma_{(j,c'',l)} \sum_{c''} \lambda_{(j,c'',l)}}{(\sum_i \sum_{c''} \lambda_{(i,c'',l)} + \sigma_l)^2} - \gamma_0, \quad (23)$$

that implies again the following contradiction: $\gamma_{(j,c,l)} \leq \gamma_{(j,c',l)}$. ■

Let's define a new hypothesis:

(H2) $\forall j, c, c'$, and l, $\gamma_{(j,c,l)} \neq \gamma_{(j,c',l)}$

With the help of **(H2)** we have the following corollary.

Corollary 4: Consider that **(H1)** and **(H2)** are verified. Then for each j and each l there exists a unique $c_{j,l} \in \{1, \ldots, C\}$ such that $\gamma_{(j,c_{j,l},l)} = \max_{c'}\{\gamma_{(j,c',l)}\}$. Then for j and l associated to c, $\underline{\lambda}_{(j,c_{j,l},l)} \geq 0$ and for all $c' \neq c_{j,l}$, $\underline{\lambda}_{(j,c',l)} = 0$. In this case the payoff of a player j can be rewritten in the following manner:

$$U_j(\vec{\lambda}_j, \vec{\lambda}_{-j}) = \sum_l \left[\frac{\gamma_{(j,l)} \lambda_{(j,l)}}{\sum_i \lambda_{(i,l)} + \sigma_l} \right] \quad (24)$$

$$+\gamma_0 \left[\phi_j - \sum_l \lambda_{(j,l)} \right]. \quad (25)$$

with $\vec{\lambda}_j = (\lambda_{(i,l)})_l$, $\lambda_{(i,l)} := \lambda_{(i,c_{i,l},l)}$, $\gamma_{(j,l)} := \gamma_{(j,c_{j,l},l)}$ and $\phi_j := \sum_c \phi_{(j,c)}$.

B. Characterizing **PNE** via Concave Programming, **PNE** closed form and **PNE** Uniqueness

Theorem 5: Consider that **(H1)** and **(H2)** are verified. If $\vec{\underline{\lambda}}$ is a **PNE** then $\underline{\Lambda}_l := \sum_{j,c} \underline{\lambda}_{(j,c,l)}$ is the unique solution of the following concave optimization problem:

$$\max_{(\Lambda_1,\ldots,\Lambda_L)} \sum_l \left[(J-1)\log(\Lambda_l+\sigma_l) - \frac{\sigma_l}{\Lambda_l+\sigma_l} + \Gamma_l(\sum_i \phi_i - \Lambda_l) \right], \tag{26}$$

with, for all l, $\Lambda_l \geq 0$ and $\Gamma_l = \sum_i \frac{\gamma_0}{\gamma_{i,l}}$.

Proof: From the KKT condition and from the fact that for all j, c and l, $\underline{\lambda}_{(j,c,l)} > 0$, $\underline{\lambda}_{(j,c,l)}$ is solution of

$$\left[\frac{1}{\sum_i \underline{\lambda}_{(i,l)} + \sigma_l} - \frac{\underline{\lambda}_{(j,l)}}{(\sum_i \underline{\lambda}_{(i,l)} + \sigma_l)^2} \right] - \frac{\gamma_0}{\gamma_{j,l}} \leq 0 \; \forall j,l. \tag{27}$$

Then by taking the sum over j in (27) and by defining $\Lambda_l = \sum_i \lambda_{(i,l)}$ and $\Gamma_l = \sum_i \frac{\gamma_0}{\gamma_{i,l}}$ we get:

$$\left[\frac{J-1}{\underline{\Lambda}_l + \sigma_l} + \frac{\sigma_l}{(\underline{\Lambda}_l + \sigma_l)^2} \right] - \Gamma_l \leq 0 \text{ for all } l, \tag{28}$$

which are the KKT condition of (26). ■

With the help of the previous theorem we prove the uniqueness of the Nash Equilibrium.

Theorem 6: Consider that **(H0)** and **(H1)** are verified. Let $\vec{\overline{\lambda}}$ and $\vec{\underline{\lambda}}$ two **PNE** such that $\vec{\overline{\lambda}} > 0$ and $\vec{\underline{\lambda}} > 0$. Under these assumptions, for all j and l

$$\overline{\lambda}_{(j,l)} = \underline{\lambda}_{(j,l)}.$$

Proof: From the KKT condition and from the fact that for all j, c and l, $\underline{\lambda}_{(j,c,l)} > 0$, we can deduce that for all j,l

$$\underline{\lambda}_{(j,l)} = (\sum_i \underline{\lambda}_{(i,l)} + \sigma_l) - \frac{\gamma_0}{\gamma_{(j,l)}}(\sum_i \underline{\lambda}_{(i,l)} + \sigma_l)^2. \tag{29}$$

Thus, we can see it is uniquely defined by $\sum_l \underline{\lambda}_{(i,l)}$ and $\sum_l \underline{\lambda}_{(i,l)}$ is unique because of the theorem 5. ■

In the next proposition, a closed form of the equilibrium will be given.

Proposition 7: Consider that **(H0)** and **(H1)** are verified. Consider that a **PNE** such that $\vec{\underline{\lambda}} > 0$ exists, then

$$\underline{\lambda}_{(j,l)} = X_l - \frac{\gamma_0}{\gamma_{(j,l)}} X_l^2, \tag{30}$$

with

$$X_l = \frac{J - 1 + \sqrt{(J-1)^2 + 4(\gamma_0 \sum_i \frac{1}{\gamma_{(i,l)}} \sigma_l)}}{2\gamma_0 \sum_i \frac{1}{\gamma_{(i,l)}}}. \tag{31}$$

Proof: From the KKT condition and from the fact that for all j, c and l, $\underline{\lambda}_{(j,c,l)} > 0$, we can deduce that for all j,l

$$\underline{\lambda}_{(j,l)} = (\sum_i \underline{\lambda}_{(i,l)} + \sigma_l) - \frac{\gamma_0}{\gamma_{(j,l)}}(\sum_i \underline{\lambda}_{(i,l)} + \sigma_l)^2. \tag{32}$$

By taking the sum over j in (33) we get:

$$\sum_i \underline{\lambda}_{(i,l)} + \sigma_l = J(\sum_i \underline{\lambda}_{(i,l)} + \sigma_l) - (\sum_i \frac{\gamma_0}{\gamma_{(i,l)}})(\sum_i \underline{\lambda}_{(i,l)} + \sigma_l)^2 + \sigma_l. \tag{33}$$

Also, we can see a second polynomial order appears. The only positive solution of it, is:

$$\sum_i \underline{\lambda}_{(i,l)} + \sigma_l = \frac{J - 1 + \sqrt{(J-1)^2 + 4(\gamma_0 \sum_i \frac{1}{\gamma_{(i,l)}} \sigma_l)}}{2\gamma_0 \sum_i \frac{1}{\gamma_{(i,l)}}}. \tag{34}$$

■

Corollary 8: Let for all l, X_l defined as in the proposition 7. If for all j,

$$\sum_l [X_l - \sum_l \frac{\gamma_0}{\gamma_{(j,l)}}] < \phi_j$$

then **(H0)** is satisfied.

C. Numerical Study of the Properties of the Nash Equilibrium

We suggest studying the impact of the number of player J and of σ_l on the **PNE**. In the two graphs in fig. 1 we consider the case of symmetric sources. The numerical study proves that, the strategy of a player j at equilibrium decreases in the number of players. In real terms, before a threshold, we can consider that the strategy of a player increases in the number of players since the topic of the message interests several players. After the threshold, it could be possible to observe that, since there are too many people involved in the topic, other players lose interest in the message because it has become too popular. For instance, this phenomenon can be seen in fashion, to explain how objects and clothes become a trend for a certain period, being adopted by a higher number of people, and when it reaches its climax, its popularity declines. Thus, our model does not capture this phenomenon. In the second figure, we consider that symmetric sources send messages on two News Feeds. When σ_1 increases, to send messages to the News Feed 1 is worthless. In this case, the utility of the source decreases. This is due to the fact that sources prefer to not send messages in News Feed 1 and News Feed 2. We cannot expect a sort of Braess Paradox in this case.

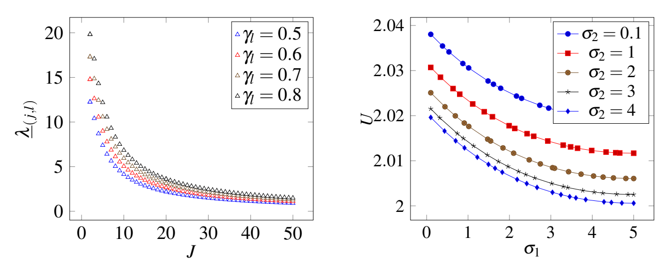

Fig. 1: Evolution of $\underline{\lambda}_{(j,l)}(J)$ for $\gamma_0 = 0.01$ and evolution of $U(\sigma_l)$ with $\gamma_1 = 0.5$, $\gamma_2 = 0.5$, $J = 5$, $\gamma_0 = 0.1$ and $\phi = 20$

After having proved the uniqueness of the Nash Equilibrium and characterizing the **PNE**, the next section will be dedicated to the learning algorithms

V. Learning Algorithm for Symmetric Sources

In this section we suggest studying a decentralized algorithm that converges to the **PNE**. It is simple to verify that $\mathbf{G}_{\vec{\sigma}}$ doesn't admit a potential function in the sense of [6],

and doesn't verify the submodularity property [16] for $J>2$ and $L>2$. Thus, we cannot use conventional tools of game theory to find out decentralized learning algorithm converges to a **PNE**. Finding a decentralized algorithm is a well-known challenging issue in the context of routing game [34]. First, with the help of the theorem 5 and the proposition 7, we are able to design a two time scale algorithm where:
- On the fast time scale, each source imitates the average behavior of sources on each News Feed each time,
- On the slow time scale, the source uses a gradient scheme to compute its optimal strategy.
We are able to prove the convergence of this decentralized algorithm to the interior **PNE**.

We assume throughout this section that **(H1)** and **(H2)** are verified. We assume that for all i, i', $\gamma_{(i,l)} = \gamma_{(i',l)} =: \gamma_l$. From **(H2)**, we restrict the learning algorithm to only one topic per News Feed. Let $n \in \mathbb{N}$ the time of update of the decentralized learning algorithm. Let $\lambda_{(j,l)}(n)$ the flow of messages sent by player j during $[n,n+1)$ on the News Feed l.

Imitation: Each source can observe a noisy version of $\sum_i \lambda(i,l)(n)$, $\overline{\Lambda}_l^j(n)$ at each instant n:

$$\overline{\Lambda}_l^j(n) = \sum_i \lambda_{(i,l)}(n) + M_{(j,l)}(n), \tag{35}$$

where $M_{(j,l)}(n)$ is, for all n, a martingale difference sequence with zero mean. The estimation scheme for $\lambda_{(j,l)}(n)$ is the following:

$$\lambda_{(j,l)}(n+1) = \left[\lambda_{(j,l)}(n) + \beta_n\left(\frac{1}{J}\overline{\Lambda}_l^j(n) - \lambda_{(j,l)}(n)\right)\right]_+, \tag{36}$$

where β_n is such that $\sum_n \beta_n = \infty$, $\sum_n \beta_n^2 < \infty$ and $[x]_+ = \max(0,x)$.

We consider that the source j knows:

$$\frac{\partial U_j}{\lambda_{(j,l)}}(\vec{\lambda}_j, \vec{\lambda}_{-j}) = \gamma_{(j,l)}\left[\frac{1}{\sum_i \lambda_{(i,l)} + \sigma_l} - \frac{\lambda_{(j,l)}}{(\sum_i \lambda_{(i,l)} + \sigma_l)^2}\right] - \gamma_0. \tag{37}$$

The source j updates the flow that it sends to a News Feed l, $\lambda_{(j,l)}$ at each instant n, at the slow timescale, with the help of a Stochastic Gradient scheme [1]:

$$\lambda_{(j,l)}(n+1) = \tag{38}$$
$$\varepsilon\left[\lambda_{(j,l)}(n) + \alpha_n\left(\frac{1}{\sum_i \lambda_{(i,l)}(n)+\sigma_l} - \frac{\lambda_{(j,l)}(n)}{(\sum_i \lambda_{(i,l)}(n)+\sigma_l)^2} - \frac{\gamma_0}{\gamma_l}\right)\right]_+,$$

with $\varepsilon > 0$, and α_n such that $\sum_n \alpha_n = \infty$, $\sum_n \alpha_n^2 < \infty$ and $\frac{\beta(n)}{\alpha(n)} \to \infty$. For instance $\alpha(n) = \frac{1}{n}$ and $\beta(n) = \frac{1}{1+n\log(n)}$ verify the previous assumption. This is a two scale algorithm. Indeed, when $\varepsilon \downarrow 0$ the stochastic gradient updates slowly compared to the imitation mechanism. In the next theorem we prove that, for all l and for all j, $\lambda_{(j,l)}(n)$ converges to $\underline{\lambda}_{(j,l)}$ by using the ODE method.

Theorem 9:

$$\lambda_{(j,l)}(n) \underset{n\to+\infty,\, \varepsilon\to 0}{\to} \underline{\lambda}_{j,l} \; a.s. \; \forall j,l \tag{39}$$

Proof: If for all j and l, we are able to prove that:

$$\lambda_{(j,l)}(t) \underset{t\to+\infty,\, \varepsilon\to 0}{\to} \underline{\lambda}_{j,l} \; \forall j,l, \tag{40}$$

with

$$\begin{aligned}\dot{\lambda}_{(j,l)}(t) = \\ \left[\tfrac{1}{J}\textstyle\sum_i \lambda_{(i,l)}(t) - \lambda_{(j,l)}(t)\right] \\ +\varepsilon\left[\frac{1}{\sum_i \lambda_{(i,l)}(t)+\sigma_l} - \frac{\lambda_{(j,l)}(t)}{(\sum_i \lambda_{(i,l)}(t)+\sigma_l)^2} - \frac{\gamma_0}{\gamma_l}\right]_+.\end{aligned} \tag{41}$$

then by using theorem 2. p.66 of [1] the theorem is proved. First let's remark that, when $\varepsilon \to 0$, we can consider for all j and l, $\lambda_{(j,l)}$ is solution of

$$\dot{\lambda}_{(j,l)}(t) = \left[\frac{1}{J}\sum_i \lambda_{(i,l)}(t) - \lambda_{(j,l)}(t)\right]. \tag{42}$$

We are in the case of cooperative ode [1] and then for all j and l the unique globally attractive rest point of this system is:

$$\frac{1}{J}\sum_i \lambda_{(i,l)} = \lambda_{(j,l)}. \tag{43}$$

The previous equation implies for all j and l,

$$\dot{\lambda}_{(j,l)}(t) = \left[\frac{1}{\sum_i \lambda_{(i,l)}(t) + \sigma_l} - \frac{\sum_i \lambda_{(i,l)}(t)}{J(\sum_i \lambda_{(i,l)}(t) + \sigma_l)^2} - \frac{\gamma_0}{\gamma_l}\right]_+ \tag{44}$$

It can be notice that (44) only depends of $\sum_i \lambda_{(i,l)}(t)$ and the rest point of (44) is the symmetric **PNE**. By taking the sum over i in (44) we obtain that $\sum_i \lambda_{(i,l)}(t)$ is solution of

$$\sum_i \dot{\lambda}_{(i,l)}(t) = \left[\frac{J}{\sum_i \lambda_{(i,l)}(t) + \sigma_l} - \frac{\sum_i \lambda_{(i,l)}(t)}{(\sum_i \lambda_{(i,l)}(t) + \sigma_l)^2} - \frac{J\gamma_0}{\gamma_l}\right]_+. \tag{45}$$

(45) converges to the $\sum_i \underline{\lambda}_{(i,l)}$. Indeed (26) can be taken as a Lyapunov function for (45) and by using the Lasalle principle the convergence can be proved (p.118, [1]). From (43) we can conclude that:

$$\lim_{t\to\infty} \lambda_{(j,l)}(t) = \lim_{t\to\infty} \frac{1}{J}\sum_i \lambda_{(i,l)}(t) = \underline{\lambda}_{(j,l)}. \tag{46}$$

■

In Fig. 5 we illustrate the convergence of (41) with two symmetric players and two News Feeds. We impose $\varepsilon = 0.01$, $\theta_1 = \theta_2 = 4$, $\frac{\gamma_1}{\gamma_0} = 0.2$ and $\frac{\gamma_2}{\gamma_0} = 0.21$. As expected, we can observe in (a) that the trajectories of $\lambda_{(1,l)}$ and $\lambda_{(2,l)}$ follow quickly the same behavior. Also, in (b) we can see the convergence of the whole system.

VI. Method for estimation of parameters with a Non-Exhaustive Data set

The purpose of this section is to show how to estimate the vector $\vec{\gamma} = (\frac{\gamma_0}{\gamma_{(i,l)}})_{(i,l)}$ using real data sets. To reach this objective we will use data extracted from Facebook. We propose to retrieve messages posts of 4 Facebook's Fan pages . These pages concern fan pages of radio stations. The fan page $p_i s_1$ is the main fan page of the radio i, $i \in \{1,2\}$, and the fan page $p_i s_2$ is the fan page of morning radio i. Both stations broadcast "pop" and are part of the top ten most listened radio stations in France. Thus, it does not seem illogical to suppose that the fan page $p_1 s_k$ competes with $p_2 s_k$ and that $p_i s_1$ and $p_i s_2$ are controlled by the same company. We extracted seven months of posts for each radio station. Our data set contains instants of arrival of messages. Fig. 2 shows the number of messages each hour over seven months and the average number of messages during a day each month. In order to estimate $\vec{\gamma}$, we initially need to

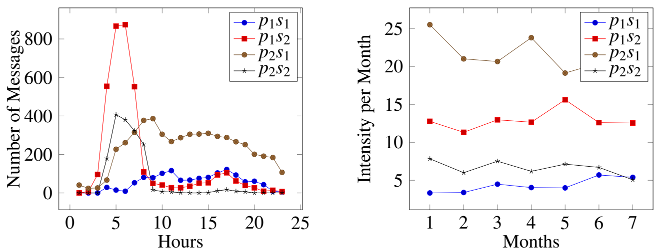

Fig. 2: Number of messages per hour and Evolution of Intensity per month

verify the assumptions of the model. The two most important hypotheses to be verified are:
- **(H3)**: Stationarity of the estimated distribution of the number of posts for each Fan Page per unit time.
- **(H3')**: The estimated intensities of each Fan Page's match, play strategies to the Nash equilibrium by Fan Page.

First, we propose to give a method to check **(H3)**. Let $N_{p_is_j}(0,24h]$ the number of messages per day per page $p_is_j$and let $P(N_{p_is_j}(0,24h] \leq x)$ the associated distribution. **(H3)** is to check whether all t, $P(N_{p_is_j}(0,24h] \leq x) = P(N_{p_is_j}(t,24h+t] \leq x)$. In order to check **(H3)** we can choose several $t \in \{1,\dots,23\}$ according to a discrete uniform law and compare them by using the test de Kolmogorov Smirnov for two samples [18] $P(N_{p_is_j}(0,24h] \leq x)$ and $P(N_{p_is_j}(t,24h+t] \leq x)$. Of course, the method can be adapted to other intervals than $(0,24h]$. In a second time, we propose a method to validate **(H3')**. We need to check if the estimated intensity, each month for instance, changes drastically. For example, we assume that a variation of the intensity less than two messages is not considered as a variation. We can observe in fig. 2 the intensity does not change more than two messages. These variations can be observed on a smaller or larger scale according to the data set.

Once we have verified **(H3)** and **(H3')**. We need to estimate the peak intervals. Peak intervals can be observe in fig. 2. A way to estimate them automatically is to use Maximum Density Algorithm [35].

We can now estimate $\vec{\gamma}$. The following proposition gives us an explicit formula for doing this:

Lemma 10: If the observed flows at equilibrium is $\vec{\lambda}$, **(H3)** and **(H3')** are verified, then for all j and l the estimated parameter $\frac{\gamma_0}{\gamma_{(i,l)}}(\vec{\lambda})$ is equal to

$$\frac{\gamma_0}{\gamma_{(i,l)}}(\vec{\lambda}) = \Big(\frac{1}{\sum_i \lambda(i,l) + \sigma_l} - \frac{\lambda_{(j,l)}}{(\sum_i \lambda(i,l) + \sigma_l)^2}\Big). \tag{47}$$

In order to estimate $\vec{\gamma}$ we proposed to assume that $\sigma_1 = \sigma_2 = 100$. In Fig. 4 we provide the estimation of the game's parameters.

Fan page	Number of Messages
p_1s_1	1271
p_1s_2	3703
p_2s_1	5321
p_2s_2	1618

Fig. 3: Number of messages per Fan Page in the studied month

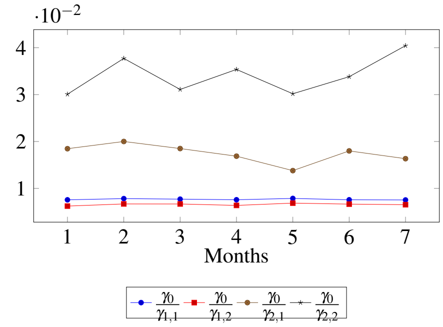

Fig. 4: Estimation of $\vec{\gamma}$ each month

VII. Discussion and Conclusion

In this paper we have defined a routing game where sources have to decide which News Feed and when they have to post, how many messages they have to post and the topic of their messages. Nash Equilibrium uniqueness, an explicit form and a characterization via concave programming of it was provided. Also a decentralized algorithm that converges to the **PNE** was proposed. Finally, a method is given to estimate parameters of the model with real data set. The empirical study applied to this model is, for us, the most promising direction for future works. Indeed, by using data of Social Network and our model we can predict the behavior of sources.

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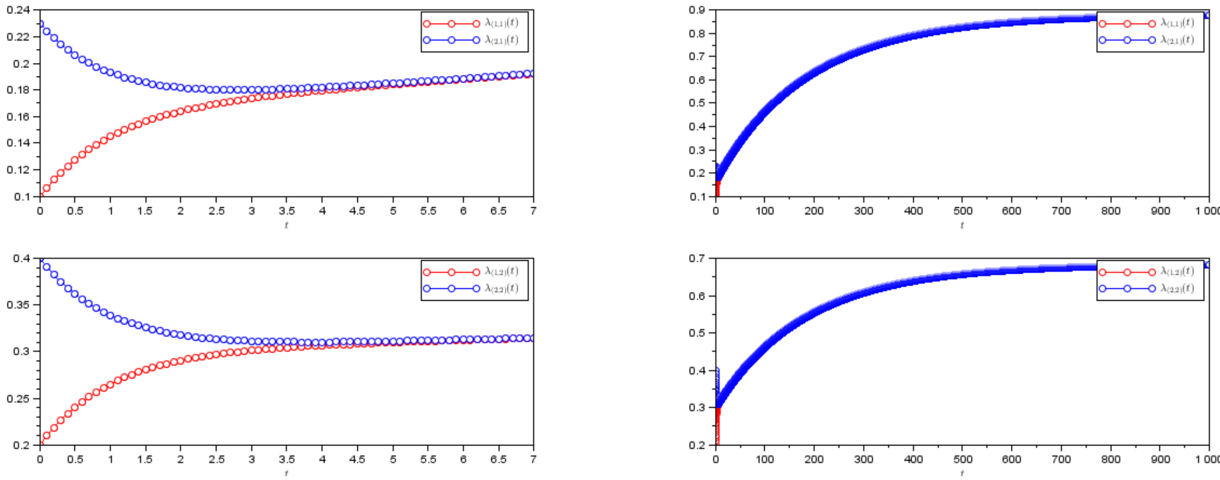

(a) (b)

Fig. 5: (a) and (b): Fast and slow evolutions, in t, of $\lambda_{(1,1)}(t)$ vs. $\lambda_{(2,1)}(t)$ and $\lambda_{(1,2)}(t)$ vs. $\lambda_{(2,2)}(t)$

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